On tadpole improvement for staggered fermions

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An explanation is proposed for the fact that Lepage–Mackenzie tadpole improvement does not work well for staggered fermions. The idea appears to work for all renormalization constants which appear in the staggered fermion self-energy. Wilson fermions are also discussed.

One-loop renormalization constants for a number of staggered-fermion operators have large finite parts (see *e.g.* ref. [1]), which appear not to be explained by gluon-tadpole contributions. This is in contrast to the situation with Wilson fermions, where tadpole improvement [2] appears to work well.

Staggered and Wilson fermions differ in the way they deal with the doubling problem: in the Wilson case, they get a mass of the order of the cutoff and decouple, while they are present in the staggered case, yielding a number of continuum flavors which is a multiple of four. The staggered doublers do contribute to loop diagrams, and here I suggest that this may explain why tadpole improvement does not work well for staggered fermions, by considering the one-loop fermion self-energy.

Let us first consider the Wilson case. The one-loop self-energy is [3]

$$\begin{aligned}\Sigma(p) \quad = \quad & \frac{4}{3}g^2\Big[-\frac{1}{8\pi^2}\int_0^1 dx\ (i\not{p}(1-x)+2m) \\ & \times\log\left(x(1-x)p^2+xm^2\right) \\ & +\sigma_0+i\not{p}\sigma_1+m\sigma_2\Big]\ ,\end{aligned} \tag{1}$$

with $\sigma_{0,1,2}$ given in Table 1 (in Feynman gauge; 1st column). The 2nd column gives the tadpole-improved values; the 3rd the contribution to those of the 2nd from the integration region $-\pi/2 < \ell_\mu \le \pi/2$ (ℓ is the loop momentum).

The table shows that gluon-tadpole improvement works well (I used the mean link in Feynman gauge). If we split up the Brillouin zone (BZ) for ℓ as $\ell=\pi_A+\tilde{\ell}$, $A=1,\ldots,16$, with $\tilde{\ell}\in(-\pi/2,\pi/2]$ and $\pi_A\in\{(0,0,0,0),(\pi,0,0,0),\ldots\}$, we see that most of the tadpole-improved values comes from the region with $\pi_A=0$. The doublers do not contribute much, since they are suppressed by the Wilson mass term.

Now, let us consider staggered fermions. The most general four-flavor mass matrix is

$$\begin{aligned}M \quad = \quad & m^S+m^V_\mu\xi_\mu+\frac{1}{2}m^T_{\mu\nu}(-i\xi_\mu\xi_\nu) \\ & +m^A_{5\mu}i\xi_\mu\xi_5+m^P_5\xi_5\ ,\end{aligned} \tag{2}$$

where the ξ_μ are a set of gamma matrices in flavor space ($m^T_{\mu\nu}$ is antisymmetric). The labels S, V, *etc.* (for scalar, vector, *etc.*) denote irreducible representations of the staggered fermion symmetry group, and correspond to zero-, one-, *etc.* link operators in the lattice action.

The staggered one-loop self-energy is [4]

$$\begin{aligned}\Sigma(p) \quad = \quad & \frac{4}{3}g^2\Big[-\frac{1}{8\pi^2}U^\dagger\int_0^1 dx\ (i\not{p}(1-x)+2M_d) \\ & \times\log\left(x(1-x)p^2+xM_d^2\right)U \\ & +\tau\ i\not{p}+\sigma_S\ m^S+\sigma_V\ m^V_\mu\xi_\mu \\ & +\frac{1}{2}\sigma_T\ m^T_{\mu\nu}(-i\xi_\mu\xi_\nu)+\sigma_A\ m^A_{5\mu}i\xi_\mu\xi_5 \\ & +\sigma_P\ m^P_5\xi_5\Big]\ ,\end{aligned} \tag{3}$$

with U diagonalizing M: $M_d=UMU^\dagger$. The values of τ and the σ's are given in Table 2 ($\overline{\sigma}_V$ is a similar constant for a one-link mass term for "reduced" staggered fermions [4]).

For the staggered case we see that gluon-tadpole improvement does not work for σ_S, σ_V and $\overline{\sigma}_V$ (it does work for σ_T, σ_A and σ_P), and that the $\pi_A=0$ region does not give the main contribution, giving a value typically much smaller than the tadpole-improved value.

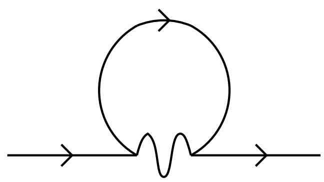
	no improvement	tadpole improved	reduced BZ
σ_0	0.326	0.0158	0.0169
σ_1	-0.0878	-0.0103	-0.00839^*
σ_2	-0.0120	-0.0120	-0.0184

Table 1
Wilson contact terms. * Depends on routing of external momentum, which can make a $\sim 25\%$ difference.

	no improvement	tadpole improved	reduced BZ
τ	-0.0446	0.0329	0.0149^*
σ_S	0.197	0.197	0.0276
σ_V	0.00385	0.0813	0.0244
σ_T	-0.117	0.0380	0.0111
σ_A	-0.209	0.0233	0.00516
σ_P	-0.294	0.0161	0.000294
$\overline{\sigma}_V$	0.0813	0.159	0.0332

Table 2
Staggered contact terms. * See Table 1.

The interpretation I would like to put forward is as follows. Consider the standard one-loop self-energy diagram:



For $\pi_A \neq 0$, the gluon propagator is very suppressed (of order a^2), and the gluon line in the diagram is effectively reduced to a four-fermion coupling between the staggered flavors. The fermion propagator, however, has poles for $\pi_A \neq 0$, and the diagram contributes "doubler-tadpoles" to the finite part of the one-loop self-energy. If we contract the gluon line to a point, the integrand near the doubler poles ($\pi_A \neq 0$) goes like $1/\tilde{\ell}^2$, and produces contributions very much like the usual gluon tadpoles: the integral is quadratically divergent ($1/a^2$), with an extra factor $\sim a^2$ coming from the contracted gluon propagator.

This idea can be tested by replacing the gluon propagator $\left[(4\sum_\mu \sin^2(\ell_\mu/2)\right]^{-1}$ (in Feynman gauge) by

$$\frac{1}{4}[P^-(\ell_1)P^+(\ell_2)P^+(\ell_3)P^+(\ell_4)+\ldots] \tag{4}$$

$$\frac{1}{8}[P^-(\ell_1)P^-(\ell_2)P^+(\ell_3)P^+(\ell_4)+\ldots]$$
$$\frac{1}{12}[P^-(\ell_1)P^-(\ell_2)P^-(\ell_3)P^+(\ell_4)+\ldots]$$
$$\frac{1}{16}P^-(\ell_1)P^-(\ell_2)P^-(\ell_3)P^-(\ell_4)\ ,$$

where

$$P^\pm(\ell_\mu) \equiv \frac{1}{2}(1 \pm \cos\ell_\mu)\ . \tag{5}$$

The factors $P^-(\ell_1)P^+(\ell_2)P^+(\ell_3)P^+(\ell_4)$, *etc.* are "smooth projectors" onto the regions $\pi_A + \tilde{\ell}$ with $\pi_A \neq 0$ of the Brillouin zone. The smoothness makes it possible to interpret this "gluon" exchange as a local four-fermion operator. The fractions 1/4, 1/8, *etc.* are the values of the real gluon propagator at $\ell = \pi_A \neq 0$. Note that the region around $\pi_A = 0$ is "projected" onto 0.

With this replacement, *i.e.* with this four-fermion interaction, one obtains the results of Table 3. We note that

- the four-fermion constants reproduce the tadpole-improved contact terms quite well, especially the larger ones;
- gluon-tadpole improvement is also needed;
- for Wilson fermions, the "four-fermion values" for σ_0, σ_1 and σ_2 are -0.00222,

	tadpole improved	four-fermion	fraction
τ	0.0329	0.0185	0.56
σ_S	0.197	0.179	0.91
σ_V	0.0813	0.0668	0.82
σ_T	0.0380	0.0309	0.81
σ_A	0.0233	0.0237	1.0
σ_P	0.0161	0.0227	1.4
$\overline{\sigma}_V$	0.159	0.134	0.84

Table 3
Staggered contact terms – comparison with four-fermion values.

-0.00111 and 0.00949, respectively, to be compared with the gluon-tadpole improved values given in Table 1.

The idea presented here can be checked on the many other staggered-fermion renormalization constants that have been calculated to one loop in perturbation theory. Also, since the approach is gauge-dependent, it should be checked for other gauges, such as Landau gauge.

Assuming that the idea is correct, one can ask how better estimates of staggered-fermion renormalization constants can be obtained. This could be done by taking the staggered-fermion theory with only the four-fermion interactions (no gluons), and computing the renormalization constants in this theory numerically. (These constants are finite, because the four-fermion interactions are irrelevant operators, proportional to g^2.)

To one-loop, one then multiplies these by the perturbatively calculated constants of the full theory (with gluons), with the four-fermion part taken out, and the appropriate power of the mean link for gluon-tadpole improvement. For example, for the wave-function renormalization, we would get

$$Z_2 = u_0^{-1} Z_2^{4\mathrm{f}} \left[1 - \frac{4}{3} g^2 \left(-\frac{1}{8\pi^2} \log a\mu + \Delta\tau\right)\right], \quad (6)$$

where u_0 is the mean link, $Z_2^{4\mathrm{f}}$ is the wave-function renormalization of the four-fermion theory, and $\Delta\tau = 0.0144$ is the difference between the first two numbers of Table 3.

This procedure resembles gluon-tadpole improvement, in that it partially resums the perturbative expansion, and it is equally heuristic. A disadvantage is that a numerical computation is needed in the four-fermion theory for each operator. A complete nonperturbative determination in the full theory [5] may therefore be preferable not only in principle, but also in practice.

A different approach would be to consider improved actions for lattice QCD with staggered fermions (see *e.g.* ref. [6]). For improved actions, one expects the couplings of high-momentum gluons and fermions to be smaller than in the unimproved case, which would presumably lead to smaller finite parts of the renormalization constants.

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